ON POLAR ORBITS OF THE EARTH'S ARTIFICIAL SATELLITES

(O POLIARNYKH ORBITAKH ISKUSTVENNYKH SPUTNIKOV ZEMLI)

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For the purpose of approximating the earth potential, [1] suggests the following potential:

$$U = \frac{fM}{2}\left\{\frac{1+i\sigma}{\sqrt{x^2+y^2+[z-c(\sigma+i)]^2}} + \frac{1-i\sigma}{\sqrt{x^2+y^2+[z-c(\sigma-i)]^2}}\right\} \qquad (1)$$

Here, x, y and z are the Cartesian coordinates of the satellite, z is the polar axis, xy is the equatorial plane, f is the coefficient of earth's attraction, M is the mass of the earth and c and σ are expressed through the values of the earth potential harmonics. The potential (1) is equivalent, within the accuracy of three terms of its expansion in a series of Legendre polynomials, to the potential of the earth and is the best of all known potentials of the earth for accuracy of approximation and simplicity of satellite motion investigation.

Reference [2] considered the polar orbits of satellites with the assumption of (1) as the approximating potential with $\sigma = 0$.

The potential (1) will be assumed in the following. Following [1] the variables x, y, z and t will be replaced by λ, μ, w and τ by means of the relationships

$$x = c\sqrt{(1+\lambda^2)(1-\mu^2)}\cos w, \qquad y = c\sqrt{(1+\lambda^2)(1-\mu^2)}\sin w, \qquad z = c\sigma + c\lambda\mu,$$
$$dt = (\lambda^2+\mu^2)\,d\tau$$

The resulting equations of motion are as follows:

$$\int_{\mu_0}^{\mu}\left[-\frac{2h}{c^2}\mu^4 + \frac{2fM\sigma}{c^3}\mu^3 + 2\left(c_2+\frac{h}{c^2}\right)\mu^2 - \frac{2fM\sigma}{c^3}\mu - (2c_2+c_1^2)\right]^{-1/2} d\mu = \tau + c_3 \qquad (2)$$

$$\int_{\lambda_0}^{\lambda} \left[\frac{2h}{c^2} \lambda^4 + \frac{2fM}{c^3} \lambda^3 + 2\left(c_2 + \frac{h}{c^2}\right)\lambda^2 + \frac{2fM}{c^3}\lambda + (2c_2 + c_1^2) \right]^{-1/2} d\lambda = \tau + c_4 \tag{3}$$

$$w = c_1 \int_{\tau_0}^{\tau} \frac{(\lambda^2 + \mu^2)\, d\tau}{(1 + \lambda^2)(1 - \mu^2)} + c_5 \tag{4}$$

where h, c_1, c_2, c_3, c_4 and c_5 are integration constants, while λ_0, μ_0 and τ_0 are some initial values for λ, μ and τ.

The orbits lying in the meridianal planes of the earth will be assumed as the polar orbits of the satellite. For such orbits it is required that $w = w_0$, i.e. it follows from (4) that $c_1 = 0$ and if c_5 is chosen zero, then $y = 0$. In this case the location of the satellite is determined by the coordinates

$$x = c\sqrt{(1 + \lambda^2)(1 - \mu^2)}, \qquad z = c\sigma + c\lambda\mu \tag{5}$$

Hence

$$\frac{x^2}{c^2(1 + \lambda^2)} + \frac{(z - c\sigma)^2}{c^2\lambda^2} = 1, \qquad \frac{x^2}{c^2(1 - \mu^2)} - \frac{(z - c\sigma)^2}{c^2\mu^2} = 1$$

Thus for λ = const there is a family of ellipses the center of which is displaced from the center of the earth by a small quantity $c\sigma$ along the z-axis, while for μ = const, there is a family of hyperbolas. It follows from this that $|\mu| \leqslant 1$, while for real satellites of the earth $c^2\lambda^2 > R^2$, with R = 6370 km (the radius of the earth). This inequality yields

$$|\lambda| > 30 \tag{6}$$

For polar orbits equations (2) and (3) for λ and μ can be rewritten as follows:

$$\left(\frac{d\mu}{d\tau}\right)^2 = \frac{2h}{c^2}(1 - \mu^2)\psi(\mu) \qquad \left(\psi(\mu) = \mu^2 - \frac{fM\sigma}{hc}\mu - \frac{c_2c^2}{h}\right) \tag{7}$$

$$\left(\frac{d\lambda}{d\tau}\right)^2 = \frac{2h}{c^2}(1 + \lambda^2)\varphi(\lambda) \qquad \left(\varphi(\lambda) = \lambda^2 + \frac{fM}{hc}\lambda + \frac{c_2c^2}{h}\right) \tag{8}$$

Let us investigate possible regions of motion. In doing this we will consider only the satellite orbits, i.e. the orbits in a limited part of space not colliding with the earth.

From the form of equations (7) and (8), it may be concluded that only those motions are possible for which the right-hand sides of equations (7) and (8) are non-negative. Let us begin with the investigation of the equation for λ. Three cases are possible.

1) The roots of the polynomial $\varphi(\lambda)$ are real and different

$$\alpha_1 = -\frac{fM}{2hc} + \sqrt{\left(\frac{fM}{2hc}\right)^2 - \frac{c_2c^2}{h}}, \qquad \alpha_2 = -\frac{fM}{2hc} - \sqrt{\left(\frac{fM}{2hc}\right)^2 - \frac{c_2c^2}{h}}$$

For $h > 0$ there must be $\lambda > \alpha_1$ or $\lambda < \alpha_2$. In both cases it appears from the geometric meaning of λ that, for $h > 0$, the motion in the present case occurs in the unlimited space (unlimited orbits). For $h < 0$ we have $\alpha_2 \leqslant \lambda \leqslant \alpha_1$.

2) In the case of equal roots $\alpha_1 = \alpha_2 = -fM/2hc$, equation (8) indicates that the motion is possible only for constant $\lambda = -fM/2hc$ if $h < 0$; if, however, $h > 0$ then the motion is possible either for the same constant λ or for any λ, the latter case again yielding unlimited orbits.

3) Let the roots of $\varphi(\lambda)$ be complex; then for $h > 0$ the motion is possible for any λ (unlimited orbits), while for $h < 0$ the motion is impossible.

In the case $h = 0$, the equation for λ is of the form

$$\left(\frac{d\lambda}{d\tau}\right)^2 = 2(1+\lambda^2)\left(\frac{fM}{c^3}\lambda + c_2\right)$$

The motion is possible for $h = 0$ if $\lambda > -c_2c^3/fM$ (unlimited orbits). And so it is clear already from the analysis of the equations for λ that for $h \geqslant 0$ the orbits in the limited space can occur only in the case of (2). It will be shown below that for $h \geqslant 0$ all orbits are unlimited. We pass to the analysis of the equations for μ.

1. The roots of the polynomial $\psi(\mu)$

$$\beta_1 = \frac{fM\sigma}{2hc} + \sqrt{\left(\frac{fM\sigma}{2hc}\right)^2 + \frac{c_2c^2}{h}}, \qquad \beta_2 = \frac{fM\sigma}{2hc} - \sqrt{\left(\frac{fM\sigma}{2hc}\right)^2 + \frac{c_2c^2}{h}}$$

are real and different; for $h > 0$ we have $\mu < \beta_2$ or $\mu > \beta_1$, but then $\beta_2 < 0$, i.e. $-1 \leqslant \mu < \beta_2 < 0 < 1$, and it follows from $\beta_2 < \beta_1$ that $-1 < \beta_2 < \beta_1 < \mu \leqslant 1$, i.e. there must be

$$|\beta_1| \leqslant 1, \qquad |\beta_2| \leqslant 1 \tag{9}$$

Since $h > 0$ for the variation of λ only in the case of (2) ($\lambda = -fM/2hc$), then it follows from (9) that

$$|\lambda| = \left|\frac{fM}{2hc}\right| \leqslant \frac{1}{|\sigma + \sqrt{1+\sigma^2}|} < 30, \qquad |\lambda| = \left|\frac{fM}{2hc}\right| \leqslant \frac{1}{|\sigma - \sqrt{1+\sigma^2}|} < 30$$

These inequalities contradict (6), i.e. for $h > 0$ in the case of (1) variation of μ the motion is impossible. For $h < 0$ there can be two variants

$$\text{(a)} \quad -1 \leqslant \mu \leqslant 1 \qquad (|\beta_1| \geqslant 1,\ |\beta_2| \geqslant 1)$$
$$\text{(b)} \quad \beta_2 \leqslant \mu \leqslant \beta_1 \qquad (|\beta_1| < 1,\ |\beta_2| < 1)$$

2. The roots of the polynomial $\psi(\mu)$ are equal $\beta_1 = \beta_2 = fM/2hc$, i.e.

$$\left(\frac{fM\sigma}{2hc}\right)^2 + \frac{c_2c^2}{h} = 0 \tag{10}$$

We note that this case is not compatible with the case (2) for λ in which

$$\left(\frac{fM}{2hc}\right)^2 - \frac{c_2c^2}{h} = 0 \tag{11}$$

Thus, for $h < 0$ there remains the case

$$\text{(c)} \quad \mu = \frac{fM\sigma}{2hc} \leqslant 1$$

The combination of the case (2) with the case (b) is also impossible since the condition $|\beta_2| < 1$ and (11) yields

$$|\lambda| = \left|\frac{fM}{2hc}\right| < \frac{1}{|\sigma + \sqrt{1+\sigma^2}|} < 30$$

which contradicts (6). The following regions of possible motions result from this analysis:

$$\text{(1a)} \quad \alpha_2 \leqslant \lambda \leqslant \alpha_1, \quad -1 \leqslant \mu \leqslant 1, \qquad \text{(1b)} \quad \alpha_2 \leqslant \lambda \leqslant \alpha_1, \quad \beta_2 \leqslant \mu \leqslant \beta_1$$
$$\text{(2a)} \quad \lambda = -\frac{fM}{2hc}, \quad -1 \leqslant \mu \leqslant 1, \qquad \text{(1c)} \quad \alpha_2 \leqslant \lambda \leqslant \alpha_1, \quad \mu = \frac{fM\sigma}{2hc} \leqslant 1$$

Figures 1 to 4 show the centers of the ellipses and hyperbolas displaced to the south by a small quantity $c\sigma$.

We will show that in the cases of (1b) and (1c) the trajectories are such that there will inevitably result a collision with the earth.

According to a theorem by Witt $|c_2c^2/h| = |\beta_1|\ |\beta_2| < 1$ but since $h < 0$ then $c_2c^2/h > 0$ for $c_2 < 0$ and $c_2c^2/h < 0$ for $c_2 > 0$. We have

$$|\alpha_2| = \left| -\frac{fM}{2hc} - \sqrt{\left(\frac{fM}{2hc}\right)^2 - \frac{c_2c^2}{h}} \right|$$
$$c_2 < 0, \qquad |\alpha_2| < \left| -\frac{fM}{2hc} - \sqrt{\left(\frac{fM}{2hc}\right)^2 - 1} \right|$$

since

$$\left(\frac{fM}{2hc}\right)^2 - 1 > \left[\left|\frac{fM}{2hc}\right| - 1\right] = \left(\frac{fM}{2hc}\right)^2 - 2\left|\frac{fM}{2hc}\right| + 1$$

Elliptic orbits. For elliptic orbits $\lambda = -fM/2hc$. Using the tables of [3] one can obtain the following expression for μ:

$$\mu = \frac{1-\beta_1+2\beta_1 \operatorname{cn}[\omega(\tau-\tau_0),k]+(1+\beta_1)\operatorname{dn}[\omega(\tau-\tau_0),k]}{\beta_1-1+2\operatorname{cn}[\omega(\tau-\tau_0),k]+(1+\beta_1)\operatorname{dn}[\omega(\tau-\tau_0),k]} = S(\tau)$$

$$\omega = \frac{\sqrt{-2h(1+\beta_1)(1-\beta_2)}}{c}, \qquad k^2 = \frac{2(\beta_1-\beta_2)}{(1+\beta_1)(1-\beta_2)}$$

Then the coordinates x and z will be determined from the expressions

$$x = a\sqrt{1-S^2(\tau)}, \qquad a = c\sqrt{1+\left(\frac{fM}{2hc}\right)^2}, \qquad z = c\sigma + bS(\tau), \quad b = -fM/2hc$$

Here a and b are major and minor semi-axes of the ellipse. The time dependence is found from the equation

$$t - t_0 = \lambda^2(\tau-\tau_0) + \frac{c}{\sqrt{-2h}}\int_{\mu_0}^{\mu}\frac{\mu^2\,d\mu}{\sqrt{(\mu^2-1)(\mu-\beta_1)(\mu-\beta_2)}} \tag{12}$$

The motion along the elliptic orbit is periodic in τ with the period $4K/\omega$ where K is the complete elliptic integral of the first kind with modulus k. The period of motion in t is found from (12).

The stability of these orbits with respect to the semi-axes and the eccentricity of the ellipse has been proved in [4].

The motion in the elliptic band. Here $\alpha_2 \leqslant \lambda \leqslant \alpha_1$ and $-1 \leqslant \mu \leqslant 1$. The quantity μ is computed in the same way as for the case of the elliptic orbit. The following expressions [2,3] are obtained for λ:

$$\lambda = \frac{A + B\operatorname{cn}[\omega_1(\tau-\tau_0),k_1]}{C + D\operatorname{cn}[\omega_1(\tau-\tau_0),k_1]}$$

$$A = \alpha_2\sqrt{1+\alpha_1^2} + \alpha_1\sqrt{1+\alpha_2^2}, \qquad C = \sqrt{1+\alpha_1^2} + \sqrt{1+\alpha_2^2}$$

$$B = \alpha_2\sqrt{1+\alpha_1^2} - \alpha_1\sqrt{1+\alpha_2^2}, \qquad D = \sqrt{1+\alpha_1^2} - \sqrt{1+\alpha_2^2}$$

$$\omega_1 = \left[\frac{4h^2}{c^4}(1+\alpha_1^2)(1+\alpha_2^2)\right]^{1/4}, \qquad k_1^2 = \frac{1}{2}\left[1 - \frac{1+\alpha_1\alpha_2}{\sqrt{(1+\alpha_1^2)(1+\alpha_2^2)}}\right]$$

The quantities x and z are computed from formulas (5).

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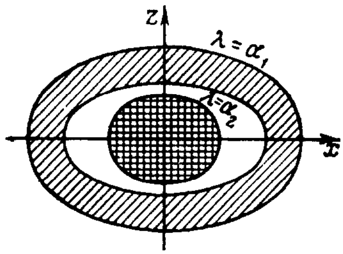

Fig. 1.

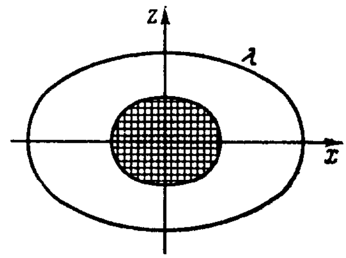

Fig. 2.

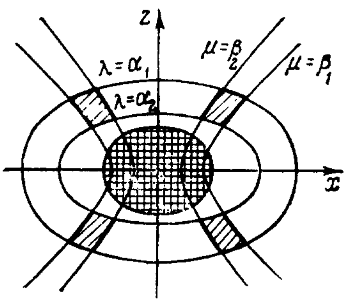

Fig. 3.

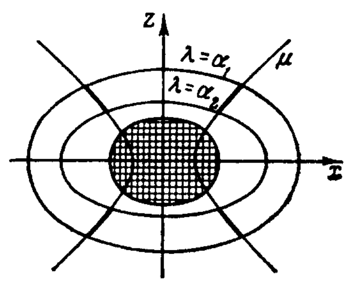

Fig. 4.

for at least $|fM/2hc| > 1$ (for $|fM/2hc| < 1$ the inequality $|\alpha_2| < 1$ is obvious). But then

$$|\alpha_2| < \left| -\frac{fM}{2hc} + \frac{fM}{2hc} + 1 \right| = 1$$

$$c_2 > 0, \qquad |\alpha_2| < \left| -\frac{fM}{2hc} - \sqrt{\left(\frac{fM}{2hc}\right)^2 + 1} \right|$$

$$\left(\frac{fM}{2hc}\right)^2 + 1 < \left(\left|\frac{fM}{2hc}\right| + 1\right)^2, \qquad |\alpha_2| < 1$$

And so $|\alpha_2| < 30$, i.e. one of the ellipses goes through the earth resulting in a certain collision with the earth (ballistic orbits).

In the case of (1c) we have, respectively

$$\mu = \frac{fM\sigma}{2hc} \leqslant 1, \qquad \left|\frac{fM}{2hc}\right| \leqslant \frac{1}{|\sigma|}, \qquad |\alpha_2| < \frac{|1 - \sqrt{1-\sigma^2}|}{|\sigma|} < 30$$

i.e. the orbit is again ballistic.

Thus, among the satellite polar orbits, only polar elliptic orbits (2a) and the trajectories in the elliptic band (1a) are possible. In the cases of (1b) and (1c) the orbits will be ballistic.

BIBLIOGRAPHY

1. Aksenov, E.P., Grebenikov, E.A. and Demin, V.G., Primenenie obobshchennoi zadachi dvukh nepodvizhnykh tsentrov v teorii dvizheniia iskusstvennykh sputnikov Zemli (Application of the generalized problem of two fixed centers in the theory of artificial earth satellite motion). *Astronom. Zh.*, No. 2, 1963.

2. Aksenov, E.P., Grebenikov, E.A. and Demin, V.G., O poliarnykh orbitakh iskusstvennykh sputnikov Zemli (On polar orbits of artificial earth satellited). *Vestn. MGU ser. fiz. astr.*, No. 5, 1962.

3. Ryzhik, I.M. and Gradshtein, I.S., *Tablitsy integralov, summ, riadov i proizvedenii (Tables of Integrals, Sums, Series and Products)*. Fizmatgiz, 1962.

4. Degtiarev, V.G., Ob ustoichivosti dvizheniia v obobshchennoi zadache dvukh nepodvizhnykh tsentrov (On the stability of motion in the generalized problem of two fixed centers). *PMM* Vol. 26, No. 6, 1962.

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